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Colliding Wave Solutions in a Symmetric Non-Metric Theory

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Abstract A method is given to generate the non-linear interaction (collision) of linearly polarized gravity coupled torsion waves in a non-metric theory. Explicit examples are given in which strong mutual focussing of gravitational waves containing impulsive and shock components coupled with torsion waves does not result in a curvature singularity. However, the collision of purely torsion waves displays a curvature singularity in the region of interaction.

Keywords Torsion waves · Colliding gravitational waves

1 Introduction

Horizon forming colliding plane wave (CPW) solutions in Einstein's general relativity including Maxwell, scalar, dilaton, axion, Yang-Mills fields and their various combinations have all been found so far [1–5]. This amounts to finding solution that instead of a curvature singularity analytically extendible Cauchy horizon forms in the interaction region of colliding waves.

Being motivated by all these examples we wish to address in this paper to the problem of whether similar type of horizon forming solutions can be found in a non-metric theory of gravity. We achieve this goal indirectly, namely by embedding particular non-metric theory into a metric one. For this purpose we start with the Eisenhart's theory of unified fields in Einstein-Cartan theory which upon reduction leads to the Einstein-scalar (ES) theory. In a series of articles, Eisenhart attempted to unify electromagnetism and gravity within the context of Einstein-Cartan theory [6, 7]. The choice of asymmetrical connection and its vanishing Ricci tensor (both symmetrical and asymmetrical parts) leads to conditions that covers a variety of Einsteinian energy-momenta.

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In this paper, we restrict ourselves entirely, through the choice of a connection to a massless scalar field which is a particular form of the Brans-Dicke-Jordan theory. From physical stand point the interesting aspect in this approach is that the possible detection of torsion waves amounts to detection of the scalar waves coupled with gravity.

In the second stage of our work, we convert solutions of colliding waves obtained in one realm (i.e. ES theory) into solutions pertaining to the other realm (i.e. non-metric theory). Hence, we choose some horizon forming solutions of colliding ES plane waves and transform them into the non-metric theory; thus by the dual interpretation we obtain solutions for colliding gravitational waves containing impulsive and shock waves coupled with torsion waves that lead to horizon forming metrics instead of singularities. The significance of the obtained solution is not restricted to colliding wave interpretation only, because the resulting metric, via the coordinate transformations can also be interpreted to represent a non-singular "distorted" Schwarzchild black-hole interior with scalar hair. We call it distorted because the inclusion of the scalar field breaks the spherical symmetry.

We also consider the collision of purely torsion waves in a non-metric theory. However, the obtained solution displays curvature singularity as the focussing hypersurface is approached. The problem of finding regular solution in the context of purely scalar waves or purely torsion waves is still open. In the literature, there are some singular CPW solutions in the non-metric theory with torsion [8] and in the scalar tensor theories without torsion [9].

Our paper is organized as follows. In Sect. 2, we give the formalism together with the main result which was obtained in Ref. [10] about the equivalence of Einstein- scalar and Brans-Dicke-Jordan theories. In Sect. 3, Einstein-scalar theory is considered. Within this context, we have obtained massless scalar field extension of the horizon forming Yurtsever (or independently Ferrari-Ibanez) solution. The regular character of the solution is emphasized by calculating the Weyl and the Ricci scalars in Appendix. Section 4, is devoted for the dual interpretation of the solution obtained in Sect. 3. This is achieved by expressing the torsion wave and non-metric tensor components. In Sect. 5, we consider the collision of purely torsion waves in a non-metric theory. The paper is concluded with a conclusion in Sect. 6.

2 The Formalism

Long time ago, it was shown that [10], Eisenhart's generalized asymmetrical connection whose empty-space equations reproduce Einstein-Maxwell field equations can be extended to cover the massless ES theory. The main concern in such studies is to introduce torsion to the background Riemann geometry by relaxing the metricity condition,

$$\nabla_\mu g_{\alpha\beta} = 0. \tag{1}$$

The geometry in which the torsion is introduced is known as Einstein-Cartan geometry defined by [11],

$$\nabla_\mu g_{\alpha\beta} = -Q_{\mu\alpha\beta}, \tag{2}$$

where ∇_μ represents the covariant derivative with respect to the asymmetrical connection, and $Q_{\mu\alpha\beta}$ is a tensor which measures the non-metricity with the property that $Q_{\mu\alpha\beta} = Q_{\mu\beta\alpha}$. The general asymmetrical connection which is derived from (2) is,

$$\Gamma^{\varkappa}_{\mu\beta} = \left\{ {\varkappa \atop \mu\beta} \right\} + g^{\alpha\varkappa}(S_{\mu\beta\alpha} - S_{\mu\alpha\beta} - S_{\beta\alpha\mu}) + \frac{1}{2}(Q_{\mu\beta}{}^{\varkappa} + Q_{\beta\mu}{}^{\varkappa} - Q^{\varkappa}_{\mu\beta}), \tag{3}$$

where $\{ {}^{\varkappa}_{\mu\beta} \}$ stands for the Christoffel symbol. The torsion tensor $S_{\mu\beta}{}^{\rho}$ can be found by antisymmetrizing the connection given in (3) and is given by,

$$S_{\mu\beta}{}^{\rho} = \frac{1}{2}(\Gamma^{\rho}_{\mu\beta} - \Gamma^{\rho}_{\beta\mu}) = \Gamma^{\rho}_{[\mu\beta]}, \tag{4}$$

while the symmetrical component reads

$$\Gamma^{\rho}_{(\mu\beta)} = \frac{1}{2}(\Gamma^{\rho}_{\mu\beta} + \Gamma^{\rho}_{\beta\mu}). \tag{5}$$

The contortion tensor $T_{\mu\beta}{}^{\varkappa}$ can be constructed from (3) in terms of torsion and non-metric tensors as,

$$T_{\mu\beta}{}^{\varkappa} = \Gamma^{\varkappa}_{\mu\beta} - \left\{ \begin{matrix} \varkappa \\ \mu\beta \end{matrix} \right\} = S_{\mu\beta\varkappa} - S_{\mu\varkappa\beta} - S_{\beta\varkappa\mu} + \frac{1}{2}(Q_{\mu\beta\varkappa} + Q_{\beta\mu\varkappa} - Q_{\varkappa\mu\beta}) \tag{6}$$

from which the following relations easily follow,

$$S_{\mu\beta\varkappa} = \frac{1}{2}(T_{\mu\beta\varkappa} - T_{\beta\mu\varkappa}), \tag{7}$$

and

$$Q_{\mu\beta\varkappa} = T_{\mu\beta\varkappa} + T_{\mu\varkappa\beta}. \tag{8}$$

The Riemann and Ricci tensors of the generalized connection $\Gamma^{\varkappa}_{\mu\beta}$ can be obtained by using the standard definitions which are found as,

$$R_{\nu\mu\lambda}{}^{\alpha} = K_{\nu\mu\lambda}{}^{\alpha} + T_{\mu\lambda;\nu}^{\alpha} - T_{\nu\lambda;\mu}^{\alpha} + T_{\nu\rho}{}^{\alpha} T_{\mu\lambda}{}^{\rho} - T_{\mu\rho}{}^{\alpha} T_{\nu\lambda}{}^{\rho}, \tag{9}$$

$$R_{\mu\lambda} = R_{\alpha\mu\lambda}{}^{\alpha} = K_{\mu\nu} + T_{\alpha\lambda;\nu}^{\alpha} - T_{\alpha\lambda;\mu}^{\alpha} + T_{\alpha\rho}{}^{\alpha} T_{\mu\lambda}{}^{\rho} - T_{\alpha\rho}{}^{\alpha} T_{\nu\lambda}{}^{\rho}, \tag{10}$$

where $K_{\nu\mu\lambda}{}^{\alpha}$, $K_{\mu\nu}$ are the Riemann and Ricci tensors respectively and semicolon ";" is the covariant derivative with respect to the Riemanian connection. It should be noted that the obtained Ricci tensor $R_{\mu\nu}$ has an antisymmetric character such that $R_{\mu\nu} \neq R_{\nu\mu}$.

In general, we assume that the contortion tensor $T_{\mu\alpha}{}^{\beta}$ is expressed in terms of the vectors k_{μ}, l_{μ} and t_{μ} which will be defined by

$$T_{\mu\alpha}{}^{\beta} = g_{\mu\alpha} k^{\beta} + l_{\mu} \delta^{\beta}_{\alpha} + t_{\alpha} \delta^{\beta}_{\mu}. \tag{11}$$

Substituting this into (10) will yield,

$$R_{\mu\nu} = K_{\mu\lambda} + g_{\mu\lambda}\big(k^{\alpha}_{;\alpha} + k^2 + 3t \cdot k\big) + l_{\mu;\lambda} - l_{\lambda;\mu} - k_{\lambda;\mu} - 3t_{\lambda;\mu} + 3t_{\lambda} t_{\mu} - k_{\mu} k_{\lambda}, \tag{12}$$

whose symmetric and antisymmetric components become, respectively

$$\begin{aligned} R_{(\mu\nu)} = K_{\mu\lambda} + g_{\mu\lambda}\big(k^{\alpha}_{;\alpha} + k^2 + 3t \cdot k\big) - \frac{1}{2}(k_{\lambda;\mu} + k_{\mu;\lambda}) \\ - \frac{3}{2}(t_{\lambda;\mu} + t_{\mu;\lambda}) + 3t_{\lambda} t_{\mu} - k_{\mu} k_{\lambda}, \end{aligned} \tag{13}$$

$$R_{[\mu\lambda]} = l_{\mu;\lambda} - l_{\lambda;\mu} - \frac{1}{2}(k_{\lambda;\mu} - k_{\mu;\lambda}) - \frac{3}{2}(t_{\lambda;\mu} - t_{\mu;\lambda}). \tag{14}$$

Let ϕ be a scalar field and define the vector l_μ as

$$l_\mu = \phi_{,\mu} - \frac{1}{2}(3t_\mu + k_\mu), \tag{15}$$

which satisfies $R_{[\mu\lambda]} = 0$. Automatically the other vectors t_μ and k_μ are obtained from the symmetric component, with the condition $R_{(\mu\nu)} = 0$. A variety of Einstein's equations with sources can be obtained if one makes the choice as

$$(3t_\lambda + k_\lambda)_{;\mu} = 3t_\mu t_\lambda - k_\mu k_\lambda + \omega T_{\mu\lambda} + f g_{\mu\nu}. \tag{16}$$

Here ω and f are functions to be found, $T_{\mu\lambda}$ is the symmetric energy momentum tensor to be specified. If the above relation is substituted in $R_{(\mu\nu)} = 0$, with the choice of the function f as

$$\begin{aligned} f &= 3t \cdot k + k^2 + k^\alpha_{;\alpha} - \frac{\omega}{2}T, \\ T &\equiv T^\alpha_\alpha, \end{aligned} \tag{17}$$

the following result will be obtained,

$$K_{\mu\nu} = \omega\left(T_{\mu\nu} - \frac{1}{2}T g_{\mu\lambda}\right). \tag{18}$$

This is exactly the Einstein's equations with sources which can be written as

$$G_{\mu\nu} \equiv K_{\mu\nu} - \frac{1}{2}K g_{\mu\lambda} = \omega T_{\mu\nu}, \tag{19}$$

while ω becomes the coupling constant.

As a particular example let us consider the coupling of a massless scalar field ϕ. The vectors are taken as follow,

$$l_\mu = -t_\mu = \frac{1}{3}k_\mu = \sqrt{\frac{\kappa}{6}}\phi_{,\mu}, \tag{20}$$

where κ is a constant. The contortion tensor is defined by,

$$T^\beta_{\mu\nu} = \sqrt{\frac{\kappa}{6}}(3g^{\beta\gamma}\phi_{,\gamma}g_{\mu\nu} + \delta^\beta_\nu\phi_{,\mu} - \delta^\beta_\mu\phi_{,\nu}) \tag{21}$$

with the trivial constraint condition $T_{\alpha\lambda}{}^\alpha = 0$, and the non-trivial one $T_{\mu\nu}{}^\alpha{}_{;\alpha} = 0$, which is equivalent to the ES field equation $g^{\alpha\beta}\phi_{;\alpha\beta} = 0$.

Hence the Ricci tensor becomes

$$R_{\mu\nu} \equiv K_{\mu\nu} - \kappa\phi_{,\mu}\phi_{,\nu} = 0, \tag{22}$$

which yields

$$K_{\mu\nu} = \kappa\phi_{,\mu}\phi_{,\nu}, \tag{23}$$

which is the Einstein–massless scalar field equations.

According to this choice the torsion and non-metric tensors are defined as,

$$S_{\mu\beta\varkappa} = \sqrt{\frac{\kappa}{6}}(g_{\beta\varkappa}\phi_{,\mu} - g_{\mu\varkappa}\phi_{,\beta}), \tag{24}$$

$$Q_{\mu\beta\varkappa} = 2\sqrt{\frac{\kappa}{6}}(g_{\mu\beta}\phi_{,\varkappa} + g_{\beta\varkappa}\phi_{,\mu} + g_{\mu\varkappa}\phi_{,\beta}). \tag{25}$$

3 Solution For Colliding ES Waves

The adopted space-time line element in general for linearly polarized case is given in Szekeres form by,

$$ds^2 = 2e^{-M}dudv - e^{-U}(e^V dx^2 + e^{-V}dy^2). \tag{26}$$

The metric functions M, U and V are functions of the null coordinates u and v only. In Ref. [2], the field equations are derived for the problem of colliding Einstein-Maxwell-scalar waves for non-linearly polarized waves and the *M-shift* method is explained how to extend the vacuum (Einstein) or electrovacuum (Einstein-Maxwell) solutions to a vacuum (electrovacuum)-scalar solutions.

In this paper, we shall use the same field equations which is shown in the previous section that they are equivalent to the field equations of Brans-Dicke-Jordan theory for Einstein-scalar case to obtain a class of regular solutions which represents colliding gravitational waves in the symmetric, non-metric space-times with torsion.

As a requirement of the *M-shift* method, the scalar field ϕ is coupled to gravitational wave through shifting the metric function M in (13), in Ref. [2] (see Ref. [2] for details) in accordance with,

$$M \to \widetilde{M} = M + \Gamma, \tag{27}$$

where the function Γ derives from the presence of the scalar field ϕ, through the conditions

$$U_u\Gamma_u = 2\phi_u^2 \quad \text{and} \quad U_v\Gamma_v = 2\phi_v^2. \tag{28}$$

We note that throughout the paper a subscript notation implies partial derivative. The integrability condition induces the massless scalar field equation as a constraint condition,

$$2\phi_{uv} - U_u\phi_v - U_v\phi_u = 0. \tag{29}$$

The most general solution to this equation is obtained if the prolate type of coordinates (τ, σ) is used instead of the null coordinates (u, v). The relation between these coordinates are defined by,

$$\tau = \sin(au_+ + bv_+), \qquad \sigma = \sin(au_+ - bv_+), \tag{30}$$

where a, b are constants and $u_+ = u\theta(u)$, $v_+ = v\theta(v)$ with $\theta(u)$ and $\theta(v)$ are unit step functions. In terms of prolate coordinates the massless scalar field equation (29) and conditions (28) becomes,

$$(\Delta\phi_\tau)_\tau - (\delta\phi_\sigma)_\sigma = 0, \tag{31}$$

$$(\tau^2 - \sigma^2)\Gamma_\tau = 2\Delta\delta\left(\tau\phi_\tau^2 + \frac{\tau\delta}{\Delta}\phi_\sigma^2 - 2\sigma\phi_\tau\phi_\sigma\right),$$

$$(\sigma^2-\tau^2)\Gamma_\sigma = 2\Delta\delta\left(\sigma\phi_\sigma^2 + \frac{\sigma\Delta}{\delta}\phi_\tau^2 - 2\tau\phi_\tau\phi_\sigma\right), \tag{32}$$

where $\Delta = 1-\tau^2$ and $\delta = 1-\sigma^2$. The exact solution to (31) is already available in [1];

$$\phi(\tau,\sigma) = \sum_n \{a_n P_n(\tau)P_n(\sigma) + b_n Q_n(\tau)Q_n(\sigma) + c_n P_n(\tau)Q_n(\sigma) + d_n P_n(\sigma)Q_n(\tau)\}, \tag{33}$$

where P and Q are the Legendre functions of the first and second kind respectively, and a_n, b_n, c_n and d_n are arbitrary constants. The choice of scalar field $\phi(\tau,\sigma)$ is extremly important as far as the regular and physically acceptable solutions are concerned. The regular solutions will be obtained if $b_n = c_n = d_n = 0$ and $a_n \neq 0$. We choose the scalar field as,

$$\phi(\tau,\sigma) = \alpha\tau\sigma + \frac{1}{4}\beta(3\tau^2-1)(3\sigma^2-1) \tag{34}$$

where α and β are arbitrary constants. Integration of (32) yields

$$\Gamma = \alpha^2(\tau^2+\sigma^2\Delta) + \frac{9}{8}\beta^2\left[\Delta(\tau^2+\sigma^2\Delta)+\tau^2\right] - 6\alpha\beta\tau\sigma\Delta\delta. \tag{35}$$

For the problem at hand, we couple the scalar field to the Cauchy-horizon forming pure gravitational wave solution obtained long ago by Yurtsever [12] (or independently Ferrari–Ibanez [13]). This particular solution is known to be isometric to the part of interior region of the Schwarzchild black hole.

The resulting metric that describes the collision of plane impulsive waves accompanied by shock gravitational waves coupled with massless scalar field is given by

$$ds^2 = 2e^{-\widetilde{M}}dudv - e^{-U}(e^V dx^2 + e^{-V}dy^2), \tag{36}$$

where the metric functions are,

$$\begin{aligned} e^{-\widetilde{M}} &= (1+\tau)^2 e^{-\Gamma}, \\ e^{-V} &= \sqrt{\frac{\delta}{\Delta}}(1+\tau)^2, \\ e^{-U} &= \sqrt{\Delta\delta}. \end{aligned} \tag{37}$$

We have shown with this example that, it is possible to construct a class of exact colliding parallel polarized plane wave solutions in the Einstein-scalar theory. Among the others it is shown that when a particular type of scalar fields couples as an initial data to an incoming parallel polarized gravitational wave results a non-singular Cauchy-horizon in the interaction region. This regularity is clearly evident by analysing the Weyl and Ricci scalars which are given in Appendix.

Another physical interpretation of the (36) is possible if the following coordinate transformation is used. Let $\psi = au_+ + bv_+$ and $\lambda = au_+ - bv_+$, together with

$$r = 1+\sin\psi, \qquad \theta = \frac{\pi}{2} - \lambda, \qquad t = \sqrt{2}x, \qquad \varphi = \sqrt{2}y, \tag{38}$$

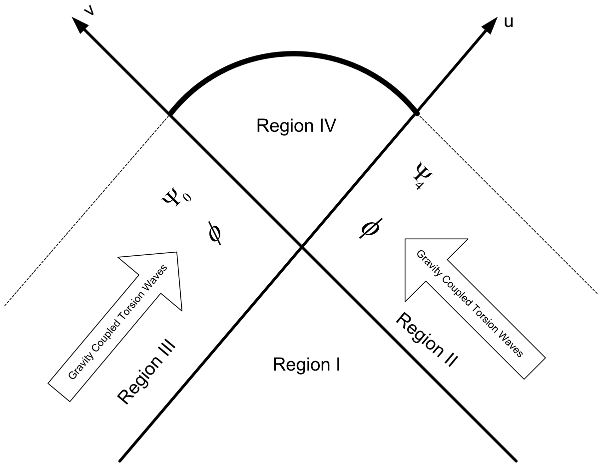

Fig. 1 The space-time diagram describes the collision of gravitational waves coupled with torsion waves

transforms the line element (36) into,

$$ds^2 = \left(1 - \frac{2}{r}\right)dt^2 - e^{-\Gamma}\left(1 - \frac{2}{r}\right)^{-1} dr^2 - r^2[e^{-\Gamma}d\theta^2 + \sin^2\theta d\varphi^2], \tag{39}$$

in which the range of the coordinate $0 \le \psi \le \frac{\pi}{2}$, confines the radial coordinate to $1 \le r \le 2$. In the absence of the scalar field, the metric (39) corresponds to the Schwarzchild black–hole interior with mass $m = 1$. With the scalar field, the metric is no more spherically symmetric and could be interpreted to represent distorted Schwarzchild black-hole interior with scalar hair.

4 Colliding Waves in a Non-Metric Theory

In Sect. 2, an analogy has been established between the Einstein-scalar and non-metric theories. As an outcome of this analogy, we consider the non-linear interaction (collision) of gravitational waves with torsion. The waves that participates in the collision are parallely polarized impulsive gravitational waves accompanied with shock gravitational waves coupled with torsion waves.

In general, the whole spacetime is divided into four continuous regions with the appropriate boundary conditions. These four regions are depicted in Fig. 1. Region I ($u < 0, v < 0$), is the flat Minkowski region: Region II ($u > 0, v < 0$) and Region III ($u < 0, v > 0$) are the plane symmetric incoming regions that contains the waves participating in the collision: Region IV ($u > 0, v > 0$) is the interaction region.

The metric and metric functions that describes the collision of parallely polarized impulsive gravitational waves accompanied with shock gravitational waves coupled with torsion waves are given in (36) and (37) respectively. The torsion waves (which is purely tensor) and the non-metric tensor components are evaluated by using the equations (24) and (25) respectively and given by

$$\begin{aligned} S_{uvu} &= \sqrt{\frac{\kappa}{6}}e^{-M}\phi_u, \qquad S_{uxx} = -\sqrt{\frac{\kappa}{6}}\frac{\Delta}{(1+\tau)^2}\phi_u, \qquad S_{uyy} = -\sqrt{\frac{\kappa}{6}}\delta(1+\tau)^2\phi_u, \\ S_{uvv} &= -\sqrt{\frac{\kappa}{6}}e^{-M}\phi_v, \qquad S_{vxx} = -\sqrt{\frac{\kappa}{6}}\frac{\Delta}{(1+\tau)^2}\phi_v, \qquad S_{vyy} = -\sqrt{\frac{\kappa}{6}}\delta(1+\tau)^2\phi_v, \end{aligned} \tag{40}$$

and

$$Q_{uuv} = 4\sqrt{\tfrac{\kappa}{6}}e^{-M}\phi_u, \qquad Q_{uxx} = -\sqrt{\frac{2\kappa}{3}}\frac{\Delta}{(1+\tau)^2}\phi_u, \qquad Q_{uyy} = -\sqrt{\frac{2\kappa}{3}}\delta(1+\tau)^2\phi_u,$$
$$Q_{vvu} = 4\sqrt{\frac{\kappa}{6}}e^{-M}\phi_v, \qquad Q_{vxx} = -\sqrt{\frac{2\kappa}{3}}\frac{\Delta}{(1+\tau)^2}\phi_v, \qquad Q_{vyy} = -\sqrt{\frac{2\kappa}{3}}\delta(1+\tau)^2\phi_v, \tag{41}$$

where ϕ_u and ϕ_v are,

$$\phi_u = a\left\{\alpha\sin 2au_+ + \frac{3}{2}\beta\left[-\frac{1}{2}(\sin 2\psi + \sin 2\lambda) + 3\sin\lambda\sin\psi\sin 2au_+\right]\right\}\theta(u),$$
$$\phi_v = b\left\{-\alpha\sin 2bv_+ + \frac{3}{2}\beta\left[\frac{1}{2}(\sin 2\lambda - \sin 2\psi) - 3\sin\lambda\sin\psi\sin 2bv_+\right]\right\}\theta(v).$$

5 Colliding Purely Torsion Waves

In this section, we consider the collision of linearly polarized purely torsion waves in a non-metric theory. The analogous problem of colliding complex and real massless scalar waves in the Einstein theory was considered long ago in the references [14] and [15] respectively. This is accomplished by taking the metric function $V = 0$ in (26) and the line element becomes

$$ds^2 = 2e^{-M}dudv - e^{-U}(dx^2 + dy^2). \tag{42}$$

This choice renders all the Weyl scalars to vanish in the incoming regions. So that the non-vanishing Ricci scalars in these regions implies purely torsion waves. Another consequence of this choice is the simplifications in the field equations describing the collision of purely torsion waves which are given by,

$$U_{uv} = U_uU_v - 2\Phi_{11}^{(0)} - 6\Lambda^{(0)}, \tag{43}$$
$$2U_{uu} = U_u^2 - 2U_uM_u + 4\phi_u^2, \tag{44}$$
$$2U_{vv} = U_v^2 - 2U_vM_v + 4\phi_v^2, \tag{45}$$
$$2M_{uv} = -U_uU_v + 4\phi_u\phi_v, \tag{46}$$

and the massless-scalar field equation which becomes equivalent to the source of the torsion waves in a non-metric theory is given by,

$$2\phi_{uv} = U_u\phi_v + U_v\phi_u. \tag{47}$$

It has been found more convinient to use prolate type coordinates in obtaining solutions to (47). Using the following transformations

$$\tau = u\sqrt{1-v^2} + v\sqrt{1-u^2},$$
$$\sigma = u\sqrt{1-v^2} - v\sqrt{1-u^2}, \tag{48}$$

the (47) transforms into (31). One of the solution to (31) is the Szekeres solution that guarantees to satisfy the boundary conditions. In prolate spheroidal coordinates this is given by,

$$\phi(u,v) = \frac{1}{2}\ln\left(\frac{1+\tau}{1-\tau}\right). \tag{49}$$

The resulting solution is obtained as,

$$
\begin{aligned}
e^{-U} &= 1 - u^2 - v^2, \\
e^{-M} &= \frac{(1-u^2-v^2)^{3/2}}{\sqrt{1-u^2}\sqrt{1-v^2}(uv+\sqrt{1-u^2}\sqrt{1-v^2})^2}.
\end{aligned}
\tag{50}
$$

The non zero torsion waves and non-metric tensor components are,

$$
\begin{aligned}
S_{uvu} &= \sqrt{\frac{\kappa}{6}}e^{-M}\phi_u, \qquad S_{uxx} = -\sqrt{\frac{\kappa}{6}}e^{-U}\phi_u, \qquad S_{uyy} = -\sqrt{\frac{\kappa}{6}}e^{-U}\phi_u, \\
S_{uvv} &= -\sqrt{\frac{\kappa}{6}}e^{-M}\phi_v, \qquad S_{vxx} = -\sqrt{\frac{\kappa}{6}}e^{-U}\phi_v, \qquad S_{vyy} = -\sqrt{\frac{\kappa}{6}}e^{-U}\phi_v,
\end{aligned}
\tag{51}
$$

and

$$
\begin{aligned}
Q_{uuv} &= 4\sqrt{\frac{\kappa}{6}}e^{-M}\phi_u, \qquad Q_{uxx} = -\sqrt{\frac{2\kappa}{3}}e^{-U}\phi_u, \qquad Q_{uyy} = -\sqrt{\frac{2\kappa}{3}}e^{-U}\phi_u, \\
Q_{vvu} &= 4\sqrt{\frac{\kappa}{6}}e^{-M}\phi_v, \qquad Q_{vxx} = -\sqrt{\frac{2\kappa}{3}}e^{-U}\phi_v, \qquad Q_{vyy} = -\sqrt{\frac{2\kappa}{3}}e^{-U}\phi_v,
\end{aligned}
\tag{52}
$$

where

$$
\begin{aligned}
\phi_u &= \frac{\theta(u)}{\sqrt{1-u^2}(\sqrt{1-u^2}\sqrt{1-v^2}-uv)} \\
\phi_v &= \frac{\theta(v)}{\sqrt{1-v^2}(\sqrt{1-u^2}\sqrt{1-v^2}-uv)}
\end{aligned}
\tag{53}
$$

We note that the null coordinates u and v are implied with a step functions $u \to u\theta(u)$ and $v \to v\theta(v)$ respectively. In contrast to the gravity coupled torsion waves, this particular example exhibits curvature singularity as the focussing hypersurface $u^2 + v^2 \to 1$ is approached. This is indicated in the scale invariant Weyl scalar $\Psi_2^{(0)}$ that arises as a result of non-linear interaction in region IV,

$$
\Psi_2^{(0)} = \frac{(\sqrt{1-u^2}\sqrt{1-v^2}+uv)^2}{\sqrt{1-u^2}\sqrt{1-v^2}(1-u^2-v^2)^2} - \frac{uv}{(1-u^2-v^2)^2}. \tag{54}
$$

The non-zero Ricci scalars are,

$$
\begin{aligned}
\Phi_{00}^{(0)} &= \frac{\theta(v)}{(1-u^2)(\sqrt{1-u^2}\sqrt{1-v^2}-uv)^2}, \\
\Phi_{22}^{(0)} &= \frac{\theta(u)}{(1-v^2)(\sqrt{1-u^2}\sqrt{1-v^2}-uv)^2}, \\
\Phi_{11}^{(0)} &= \frac{\theta(u)\theta(v)}{\sqrt{1-u^2}\sqrt{1-v^2}(\sqrt{1-u^2}\sqrt{1-v^2}-uv)^2}, \\
\Lambda^{(0)} &= -\frac{1}{3}\Phi_{11}^{(0)}.
\end{aligned}
\tag{55}
$$

6 Conclusion

In this study, we have presented two types of colliding plane wave solutions in the symmetric non-metric theory. This is accomplished by using an analogy which was developed long ago between the metric and non-metric theories. This analogy reveals the equivalence of the field equations in Einstein-scalar and Brans-Dicke-Jordan theories.

One of the obtained solution describes the collision of impulsive gravitational waves accompanied with shock gravitational waves coupled with torsion waves. This particular solution has an interesting property that, in the region of interaction, an analytically extendible Cauchy horizon forms in place of a curvature singularity. On the other hand, the collision of purely torsion waves results in a curvatuıre singularity in the interaction region.

Appendix

The non-zero Weyl and Ricci scalars for the collision of parallely polarized impulsive gravitational waves accompanied with shock gravitational waves coupled with massless-scalar wave are obtained as follows.

$$\begin{aligned}\Psi_2 = {}& \frac{9abe^{\Gamma}}{8(1+\sin\psi)^3}\left\{\frac{\beta^2\cos^2\psi}{3}[\cos^2\psi + 3\cos^2\lambda(\cos 2\psi - 1)](1+\sin\psi)\right. \\ & \left. + \frac{4}{9}\alpha(1+\sin\psi)(\cos 2\psi - \cos 2\lambda)\left[2\beta\sin\psi\sin\lambda + \frac{\alpha}{3}\right] + \frac{8}{9}\right\}\theta(u)\theta(v), \end{aligned} \tag{56}$$

$$\begin{aligned}e^{-\Gamma}\Psi_0 = {}& \frac{b}{(1+\sin au)^2\cos au}\delta(v) + \frac{2b^2\theta(v)}{(1+\sin\psi)^2}\left\{\frac{9\beta^2\cos^2\psi}{16}[\cos^2\psi(3\cos^2\lambda - 1)\right. \\ & + \cos\psi\cos\lambda\sin\lambda(3\sin\psi - 2) + 2\cos^2\lambda(2\sin\psi - 1)] \\ & + \frac{3}{2}\alpha\beta\{\cos\psi\cos\lambda[4\cos^2\psi(6\cos^2\lambda - 5) + \sin\psi(6\cos^2\lambda - 4) - 5\cos^2\lambda] \\ & + 2\sin\lambda\cos^2\lambda[3\cos^2\psi(\sin\psi - 1) - \sin\psi + 2] - 2\sin\psi\sin\lambda\cos^2\psi\} \\ & + \frac{\alpha^2}{2}[2\cos\psi\sin\lambda\cos\lambda(\sin P - 1) + \cos^2\lambda(2\sin\psi - 1) + \cos^2\psi(2\cos^2\lambda - 1)] \\ & \left. - \frac{3}{2}(1+\sin\psi)^{-1}\right\}, \end{aligned} \tag{57}$$

$$\begin{aligned}e^{-\Gamma}\Psi_4 = {}& \frac{a}{(1+\sin bv)^2\cos bv}\delta(u) - \frac{2a^2\theta(u)}{(1+\sin\psi)^2}\left\{\frac{9\beta^2\cos^2\psi}{16}[\cos^2\psi(1 - 3\cos^2\lambda)\right. \\ & + \cos\psi\cos\lambda\sin\lambda(3\sin\psi - 2) + 2\cos^2\lambda(1 - 2\sin\psi)] \\ & + \frac{3}{2}\alpha\beta\{\cos\psi\cos\lambda[4 + \cos^2\psi(6\cos^2\lambda - 5) + \sin\psi(6\cos^2\lambda - 4) - 5\cos^2\lambda] \\ & + 2\sin\lambda\cos^2\lambda[3\cos^2\psi(1 - \sin\psi) + \sin\psi - 2] + 2\sin\psi\sin\lambda\cos^2\psi\} \\ & + \frac{\alpha^2}{2}[2\cos\psi\sin\lambda\cos\lambda(\sin\psi - 1) + \cos^2\lambda(1 - 2\sin\psi) + \cos^2\psi(1 - 2\cos^2\lambda)] \\ & \left. + \frac{3}{2}(1+\sin\psi)^{-1}\right\}, \end{aligned} \tag{58}$$

$$e^{-\Gamma}\Phi_{22} = \frac{2a^2}{(1+\sin\psi)^2}\left\{\frac{9\beta^2\cos^2\psi\sin 2au}{16}[\sin 2au + \sin\psi\cos\lambda]\right.$$
$$+\frac{3}{2}\alpha\beta\left\{2\cos 2au[\cos^2\psi(2\cos^2\lambda-1)-\cos^2\lambda\sin^2\psi]\right.$$
$$\left.\left.+3\cos\psi\cos\lambda\left(\frac{4}{3}-\cos^2\lambda-\cos^2\psi\right)\right\}+\frac{\alpha^2}{2}\sin^2 2au\right\} \tag{59}$$

$$e^{-\Gamma}\Phi_{00} = \frac{2b^2}{(1+\sin\psi)^2}\left\{\frac{9\beta^2\cos^2\psi\sin 2bv}{16}[\sin 2bv + \sin\psi\cos\lambda]\right.$$
$$+\frac{3}{2}\alpha\beta\left\{2\cos 2bv[\cos^2\lambda(2\cos^2\psi-1)-\cos^2\psi\sin^2\lambda]\right.$$
$$\left.\left.+3\cos\psi\cos\lambda\left(\frac{4}{3}-\cos^2\lambda-\cos^2\psi\right)\right\}+\frac{\alpha^2}{2}\sin^2 2au\right\}, \tag{60}$$

$$\Phi_{02} = 0, \tag{61}$$

$$\Phi_{11} = -3\Lambda = \frac{abe^{\Gamma}}{16(1+\sin\psi)^2}\{\beta^2\{9\cos^2\psi[3\cos^2\lambda(\cos 2\psi - 1)+\cos^2\psi]\}$$
$$-4\alpha(\cos 2\lambda - \cos 2\psi)(6\beta\sin\psi\sin\lambda+\alpha)\}\theta(u)\theta(v). \tag{62}$$

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